Generalized error propagation in one-dimensional chaotic systems

G. Malescio
Dipartimento di Fisica, Università di Messina, I-98100 Messina, Italy
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The finiteness of observational and computational precision suggests that not only the initial condition, as is usually assumed, but also the evolution law of dynamical systems is affected by unavoidable uncertainties. The consequences are explored for chaotic systems by suitably generalizing the concept of the Lyapunov exponent in the case of one-dimensional maps. The relation between the results obtained and the shadowing problem is discussed.

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I. INTRODUCTION

A fundamental feature of chaotic systems is the so-called "sensitive dependence on initial conditions": due to the exponential growth of initial errors, different orbits starting close together will move rapidly apart. This behavior dramatically points out the impossibility of measuring or computing anything (not only initial conditions) exactly [1]. In spite of this, dynamical systems are generally studied in the hypothesis that the law governing their time evolution (i.e., the force acting on the system) is exactly defined. It follows that error influence is underestimated, a fact which can be of extreme relevance in the case of exponential propagation of errors.

According to the above considerations, when modeling physical systems, one should assume that the evolution law of the system considered is only approximately well defined or, in a less radical approach, that it has a given functional form, but depending on parameters which cannot be exactly defined. For example, referring to a typical chaotic system as the Duffing oscillator (a damped, nonlinear, sinusoidally forced oscillator), it should be taken into account that uncertainties affect unavoidably not only the definition of the position and the velocity of the point mass, as is currently assumed, but also that of the elastic constant, of the damping coefficient, and of the amplitude and frequency of the external force. In this paper we focus our attention on the simplest class of dynamical systems able to exhibit a chaotic behavior, i.e., noninvertible one-dimensional (1D) maps. Consistently with the preceding remarks, error propagation in these systems will be studied in the hypothesis that errors affect both the initial condition and the map itself.

II. GENERALIZED LYAPUNOV EXPONENT FOR 1D MAPS

One-dimensional maps can be written in the general form

$$x_n = f(x_{n-1}) , \tag{1}$$

with $f(x): I \to I$, where $I \in \mathbb{R}$ is some bounded interval on the real line $\mathbb{R}$. Mathematically one defines as true orbit $\{x_n\}$ that one which satisfies Eq. (1) while the term pseudo-orbit is used to describe orbits which arise upon the introduction of noise or when f is only approximately well defined. More precisely one calls $\{p_n\}_{n=n_1,n_2}$ an ϵ pseudo-orbit for f if $|p_{n+1} - f(p_n)| < \epsilon$ for all $n_1 \le n \le n_2$. Moreover, given a pseudo-orbit $\{p_n\}$, one says that a true orbit $\{x_n\}$ δ-shadows $\{p_n\}$ on $[n_1, n_2]$ if $|x_n - p_n| < \delta$ for all $n_1 \le n \le n_2$. That pseudo-orbits can be shadowed by true orbits has been rigorously proved under given assumptions for $f(x)$ [2,3]. While we refer the interested reader to the quoted references for an exact statement of shadowing results, these say essentially that for hyperbolic and structurally stable systems any ϵ perturbation of the map can be *counteracted* by a change in the initial condition so that the new true orbit is ϵ close to the original trajectory. More recently it has been proved that a "mild" form of the shadowing property holds also for chaotic, nonhyperbolic systems: while the noisy orbit rapidly diverges from the true orbit with the same initial point, there exists a different true orbit with a slightly different initial point which stays near the noisy orbit for a long time [4]. To the author's knowledge these mathematical results seem not to have been considered in relation to the error propagation process in physical systems. On the other hand, it is obvious that, taking into due account the unavoidable finiteness of observational and computational precision, one cannot distinguish between true and pseudo-orbits: uncertainties affect both initial conditions and the law governing the system time evolution, i.e., the function $f(x)$, and thus one has to consider their combined effects on the system evolution.

For 1D maps, as well as for more complicated systems, orbital divergence or convergence may be conveniently expressed by Lyapunov exponents. These usually quantify the (exponential) amplification, or reduction, of the error on the initial condition. In order to generalize the concept of the Lyapunov exponent to the case of an approximately well-defined f, we add to f a stochastic perturbation which is arbitrarily time and x dependent and consider the propagation of errors through the map with a linear stability analysis [5]. If dx_{n-1} is the error in specifying x_{n-1} then

$$x_n + dx_n = f(x_{n-1} + dx_{n-1}) + \epsilon R_{n-1}$$

$$\simeq f(x_{n-1}) + dx_{n-1} \frac{df}{dx}\bigg|_{x_{n-1}} + \epsilon R_{n-1} , \tag{2}$$

where ϵ is a small quantity and R_{n-1} is a random real number varying on a limited interval $[R_{min}, R_{max}]$; moreover it has been supposed that $f(x)$ is analytic in I except for a negligible set of points. The resulting uncertainty on x_n is

$$dx_n = dx_{n-1}\frac{df}{dx}\bigg|_{x_{n-1}} + \epsilon R_{n-1} , \tag{3}$$

which, in terms of the error on the initial point x_0, becomes

$$\begin{aligned} dx_n &= dx_0 \prod_{i=0}^{n-1} \frac{df}{dx}\bigg|_{x_i} \\ &\quad + \epsilon \left\{ \sum_{i=0}^{n-2} R_i \left[\prod_{j=i+1}^{n-1} \frac{df}{dx}\bigg|_{x_j} \right] + R_{n-1} \right\} \\ &= (dx_n)_x + (dx_n)_f . \end{aligned} \tag{4}$$

Thus the error on x_n is the sum of two contributions: the first one, $(dx_n)_x$, is well known and derives from the uncertainty on the initial value of the variable x; the second one, $(dx_n)_f$, arises from the uncertainty on the map itself. If both contributions are to grow exponentially (so long as the orbit separation remains infinitesimal), these can be written, respectively, in the form $(dx_n)_x = dx_0 2^{n\lambda_x}$, $(dx_n)_f = \epsilon 2^{n\lambda_f}$ and Eq. (4) becomes

$$dx_n = dx_0 2^{n\lambda_x} + \epsilon 2^{n\lambda_f} . \tag{5}$$

Here λ_x is the standard Lyapunov exponent [5,6]:

$$\lambda_x = \lim_{n\to\infty} \left[\frac{1}{n}\right] \log_2 \left| \prod_{i=0}^{n-1} \frac{df}{dx}\bigg|_{x_i} \right| , \tag{6}$$

while λ_f is a generalized Lyapunov exponent defined as

$$\lambda_f = \lim_{n\to\infty} (1/n) \log_2 \Lambda , \tag{7}$$

where $\Lambda = |\sum_{i=0}^{n-1} R_i a_i|$ with $a_i = \prod_{j=i+1}^{n-1} (df/dx|_{x_j})$ for $0 < i < n-1$ and $a_{n-1} = 1$. The limit of large n is necessary in order to obtain a quantity that describes long-term behavior and is independent of the initial conditions. In the notation introduced $\lambda_x = \lim_{n\to\infty} (1/n) \log_2 |a_0|$, thus Eq. (7) can be written also as

$$\lambda_f = \lambda_x + \Delta\lambda , \tag{8}$$

where $\Delta\lambda = \lim_{n\to\infty} (1/n) \log_2 (\Lambda/|a_0|)$ and we have supposed $a_0 \neq 0$ (thus excluding superstable points).

For later purposes we note that Eq. (6) implies

$$\lim_{n\to\infty} \left| \prod_{i=0}^{n-1} \frac{df}{dx}\bigg|_{x_i} \right| = (2^{\lambda_x})^n , \tag{9}$$

which, since the definition of λ_x should be independent of the initial point for nearly all x, can be generalized as

$$\lim_{n_2 - n_1 \to\infty} \left| \prod_{i=n_1}^{n_2} \frac{df}{dx}\bigg|_{x_i} \right| = (2^{\lambda_x})^{n_2 - n_1 + 1}, \quad n_2 > n_1 . \tag{10}$$

We further observe that the form $\lim_{n\to\infty} (1/n) \log_a A$ (with $a > 1$ and A definite positive) is positive (negative) if for large n the quantity A grows (goes to zero) exponentially or faster, whereas it is zero if A grows or goes to zero slower than exponentially, and of course if A tends to a finite nonzero value.

III. ANALYSIS AND RESULTS

As is well known [5,6], positive values of λ_x indicate orbital divergence and chaos whereas negative values are typical of ordered motions and set the time scale on which perturbations of the systems's state will decay. Our aim is to study the behavior of λ_f and to relate it to that of λ_x. We first show that λ_f cannot be negative, or in other words that the error on x_n deriving from the uncertainty on f is never (exponentially) reduced. In order that λ_f be negative, for large n the quantity Λ should go to zero exponentially or faster, so let us suppose that

$$\lim_{n\to\infty} \Lambda = \lim_{n\to\infty} \left| \sum_{i=0}^{n-1} R_i a_i \right| \le b^{-n} ,$$

with $b > 1$; then for large n

$$\left| \sum_{i=0}^{n} R_i a_i \right| = \left| R_n + \left[\sum_{i=0}^{n-1} R_i a_i \right] \frac{df}{dx}\bigg|_{x_n} \right| \le b^{-(n+1)} . \tag{11}$$

Since $df/dx|_{x_n}$ is a limited quantity, the second term of the first member of the above inequality tends to zero for $n \to \infty$ while R_n is in general different from zero except for some n (provided that the zero is included in the interval $[R_{min}, R_{max}]$). It follows that Eq. (11) cannot be satisfied and thus λ_f cannot be negative.

Let us now investigate if λ_f can be positive, i.e., if Λ can grow exponentially or faster. We consider first the ordered regime. Splitting the sum, Λ can be written as

$$\Lambda = \left| \sum_{i=0}^{n-m} R_i a_i + \sum_{i=n-m+1}^{n-1} R_i a_i \right| , \tag{12}$$

where m is a finite integer sufficiently great that for $i \le n-m$ it is possible to assume, according to Eq. (10), $|a_i| \simeq (2^{\lambda_x})^{n-i-1}$. The first sum in Eq. (12) can then be maximized as follows:

$$\begin{aligned} \sum_{i=0}^{n-m} R_i a_i \le M \sum_{i=0}^{n-m} |a_i| &\simeq M \sum_{i=0}^{n-m} (2^{\lambda_x})^{n-i-1} \\ &= M 2^{-\lambda_x} \sum_{k=m}^{n} (2^{\lambda_x})^k , \end{aligned} \tag{13}$$

where $M = |R_{max}|$. For ordered motions $\lambda_x < 0$ (with the exclusion of bifurcation points where $\lambda_x = 0$), so $2^{\lambda_x} < 1$ and the sum in Eq. (13) is convergent. The second sum in Eq. (12) is made up of a finite number of terms, each one being the product of a finite number of limited terms, thus the sum itself is finite. It follows that in the ordered regime, since Λ is a finite quantity, λ_f cannot be positive and, being also non-negative, it must be zero.

Let us now consider the chaotic regime. In this case since $\lambda_x > 0$ the final sum in Eq. (13) would diverge so, as will soon be clear, it is more convenient to analyze the behavior of the quantity $\Lambda/|a_0|$ which we write, splitting the sum, as

$$\frac{\Lambda}{|a_0|} = \left| \sum_{i=0}^{m-1} \frac{R_i a_i}{a_0} + \sum_{i=m}^{n-1} \frac{R_i a_i}{a_0} \right| . \tag{14}$$

Here m is a finite integer sufficiently great that for $i > m$ it is possible to assume

$$\left| \frac{a_i}{a_0} \right| = \left| \frac{1}{\prod_{j=1}^{i} (df/dx|_{x_j})} \right| \simeq (2^{-\lambda_x})^i$$

as follows from Eq. (10). The first sum in Eq. (14) is finite since it is made up of a finite number of limited terms. Maximizing the second sum we have in the limit of large n

$$\sum_{i=m}^{n-1} \frac{R_i a_i}{a_0} \le M \sum_{i=m}^{n-1} \left| \frac{a_i}{a_0} \right| \simeq M \sum_{i=m}^{n-1} (2^{-\lambda_x})^i . \tag{15}$$

Since $2^{-\lambda_x} < 1$ the sum in Eq. (15) is convergent and consequently $\Lambda/|a_0|$ is a finite quantity. Moreover $\Lambda/|a_0|$ cannot go to zero faster than exponentially: in fact in this case $\Delta\lambda$ and thus also λ_f (in the hypothesis of a finite λ_x) would tend to $-\infty$ in contrast with the general result obtained at the beginning of this section. Then if we prove that $\Lambda/|a_0|$ cannot go to zero exponentially it will follow that $\Delta\lambda = 0$ and consequently $\lambda_f = \lambda_x$. Let us suppose that $\lim_{n\to\infty} \Lambda/|a_0| = b^{-n}$ with $b > 1$. Then for large n we can write

$$\left| \sum_{i=0}^{n} \frac{R_i a_i}{a_0} \right| = \left| \sum_{i=0}^{n-1} \frac{R_i a_i}{a_0} + \frac{R_n a_n}{a_0} \right| = |s b^{-n} + s_n R_n (2^{-\lambda_x})^n| = b^{-(n+1)} , \tag{16}$$

where $s = \mathrm{sgn}(\sum_{i=0}^{n-1} R_i a_i/a_0)$, $s_n = \mathrm{sgn}(a_n/a_0)$ and, making use of Eq. (10), we wrote a_n/a_0 as $s_n(2^{-\lambda_x})^n$. Multiplying Eq. (16) for $b^{(n+1)}$ we obtain

$$b|s + s_n R_n (b 2^{-\lambda_x})^n| = 1 . \tag{17}$$

First of all we note that a necessary condition for satisfying Eq. (17) in the limit of large n is that $b2^{-\lambda_x} = 1$. In fact, since R_n is in general different from zero, $b2^{-\lambda_x}$ cannot be greater than one otherwise the left member of Eq. (17) would grow exponentially; on the other hand, $b2^{-\lambda_x}$ cannot be smaller than one because since R_n is finite the left member would tend to b which is strictly greater than one. Then Eq. (17) amounts to

$$|s + s_n R_n| = b^{-1} . \tag{18}$$

It is evident that since R_n is a random number Eq. (18) cannot in general be satisfied except at most for some n and thus $\Lambda/|a_0|$ cannot go to zero exponentially for large n.

Let us analyze the consequences of the above results. In the ordered regime λ_f is zero so the separation of the orbits due to the error on $f(x)$ is not exponentially reduced as that deriving from the uncertainty on the initial conditions and consequently strict orbital convergence cannot be attained, save in the limit $\epsilon \to 0$. On the other hand, since $(dx_n)_f$ is neither exponentially amplified, the perturbed orbit is ϵ close to the unperturbed one for every n and thus the shadowing property is satisfied. It is to be stressed that these considerations are valid only as long as $\lambda_f = 0$: as already remarked this cannot be proved for bifurcation points, which are nonhyperbolic points [7]. In the chaotic regime λ_f is positive and its value is identical to that of λ_x, so the contribution to the final error deriving from the uncertainty on the initial condition and that originated by the error on $f(x)$ grow exponentially and with the same rate. This is essential in order that shadowing be possible. In fact, for a pseudo-orbit starting from x_0 the error at the nth iteration is $(dx_n)_f = \epsilon 2^{n\lambda_f}$; on the other hand, for a true orbit starting at $x_0 + dx_0$ the error at the nth iteration is $(dx_n)_x = dx_0 2^{n\lambda_x}$. It is obvious that the true orbit can shadow the noisy one only if these two quantities grow exactly with the same rate.

The results obtained are quite general and hold for all 1D maps under the very mild assumptions made. In the following we report numerical results for two of the most representative examples of chaotic 1D maps. Let us first consider the so-called tent map:

$$f(x_n) = \mu(1 - 2|x_n - 0.5|) , \tag{19}$$

where, x being restricted to the interval [0,1], it is required that $0 \le \mu \le 1$. If $\mu < 0.5$, for every initial value x_0 the iterates tend asymptotically to the fixed point $x = 0$, whereas for $\mu > 0.5$ the orbit of a single initial point appears chaotically distributed all over a point set dense in a finite interval which goes to (0,1) as $\mu \to 1$. For f given by Eq. (19) $|(df/dx|_{x_i})| = 2\mu$ for every i, thus the standard Lyapunov exponent is simply $\lambda_x = \log_2 2\mu$. In agreement with the nature of the attractor, this quantity is negative for $\mu < 0.5$ (ordered regime) and positive for $\mu > 0.5$ (chaotic regime), tending to one for $\mu \to 1$. In Fig. 1 the generalized Lyapunov exponent λ_f is shown as a

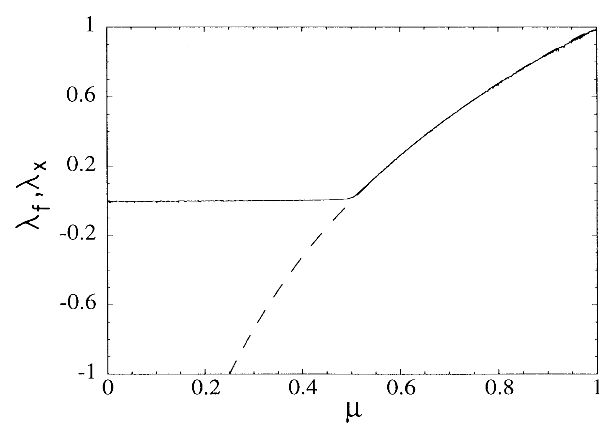

FIG. 1. λ_f (solid line) and λ_x (dashed line) as a function of μ for the tent map.

function of μ and compared with λ_x. As predicted by the preceding analysis, λ_f is zero for ordered motions, whereas it is positive and nearly indistinguishable from λ_x in the chaotic region. The ripples shown by λ_f are due to the stochastic nature of the error on $f(x)$ and can be reduced by improving numerical accuracy in the evaluation of Eq. (7).

We next consider the logistic map

$$f(x_n)=\mu x_n(1-x_n) , \qquad (20)$$

with x varying in the interval [0,1] and $0 \leq \mu \leq 4$. The behavior of the logistic map is an example of the period doubling route to chaos [8]: initially the attracting set consists of a single point that bifurcates into a two-point cycle at $\mu=3$; subsequently this bifurcates into a four-point cycle and so on, until at $\mu_\infty \simeq 3.57$ a cycle of infinite length, corresponding to a chaotic attractor, appears; the chaotic region between μ_∞ and 4 is, however, interspersed with small "windows" where the attracting set is again a periodic cycle. In Fig. 2 λ_f is shown as a function of μ; in complete agreement with our general analysis, λ_f is zero (within numerical accuracy) for $\mu < \mu_\infty$, whereas for $\mu > \mu_\infty$ it exhibits a trend towards an increasingly chaotic behavior (with numerical value quite similar to that of λ_x) interrupted by the "windows" of periodic behavior in which λ_f is again zero.

A graphical analysis, shown in Fig. 3, makes it possible to illustrate the orbital divergence process. The error bar on some experimentally determined single initial condition is represented by two nearby initial conditions. Similarly the uncertainty affecting the map is represented by drawing two logistic maps corresponding to slightly different values of μ. Let us first consider the trajectory starting from one of the initial points. Upon the first iteration, due to the uncertainty on the map, an error bar is generated on the trajectory. This could be represented by drawing two trajectories corresponding to the iteration of the incoming one by each of the two nearby maps.

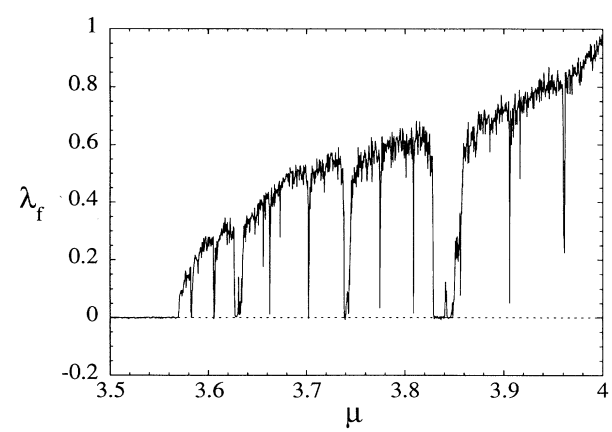

FIG. 2. λ_f as a function of μ for the logistic map; the figure details the region $3.5 \leq \mu \leq 4$. For $\mu \leq 3.5$ the curve, which is everywhere zero within numerical accuracy, is not shown.

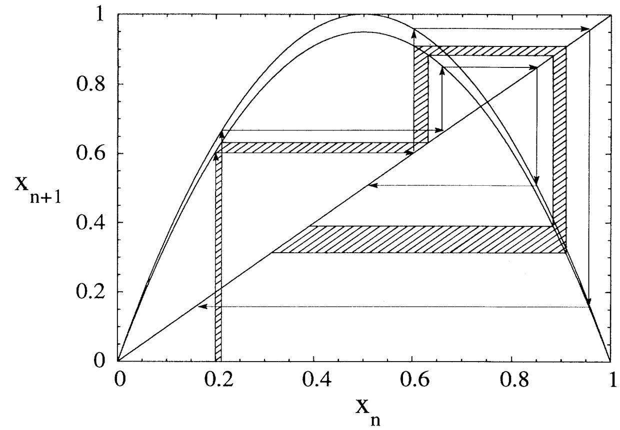

FIG. 3. Successive iterations through the logistic map for $\mu \lesssim 4$; the dashed area shows the separation of the orbits for $\epsilon=0$ (see text for explanations).

In a pictorial language we could say that the single trajectory is transformed into a "beam" of possible trajectories. Each of these is in turn transformed upon the successive iteration into another beam of trajectories. The process is repeated at each iteration. Thus a progressive widening of the resulting overall beam, i.e., an increasing uncertainty on x_n, follows even for an exactly defined initial condition. Actually, due to the error on the initial condition, we have an initial beam of possible trajectories each one undergoing the above-described process. As a consequence the overall orbital divergence is faster, as pointed out in the figure, than that corresponding to the usual hypothesis of an exactly defined μ. If it is assumed that the initial condition x_0 and the function $f(x)$ can be specified to the same degree of accuracy, i.e., if $dx_0 \simeq \epsilon = \Delta$, Eq. (5) can be written as

$$dx_n \simeq \Delta(2^{n\lambda_x}+2^{n\lambda_f}) , \qquad (21)$$

which in the chaotic regime, being $\lambda_x=\lambda_f=\lambda$, yields $dx_n \simeq 2\Delta 2^{n\lambda}$. Thus at each iteration the error on the variable x is twice greater than that estimated in the hypothesis of an exactly defined $f(x)$ and in the fully chaotic case ($\lambda \simeq 1.0$ bit per iteration [9]) $dx_n \simeq 2\Delta 2^n = \Delta 2^{n+1}$ which implies that predictive power is completely lost one iteration before.

In conclusion, in this paper we pointed out that, due to the finitess of observational and computational precision, error influence cannot be restricted to initial conditions only, as currently assumed in the study of dynamical systems. This consideration led us to investigate error propagation in one-dimensional maps in the hypothesis that errors affect both the initial condition x_0 and the very correspondence between x_n and x_{n+1}. The results obtained show that both kinds of uncertainties contribute to the final error. It follows that a faster orbit separation is to be expected in the chaotic regime, where both contributions grow exponentially and with the same rate (a necessary condition in order that the shadowing property hold). Moreover, even for ordered motions strict orbital convergence cannot in general be attained. In summary a reduced overall dynamical stability follows from a less restricted and, according to the exposed considerations, more realistic evaluation of error influence. The analysis presented will be extended in the immediate future to more complicated systems.

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